SOME NOTES re: Matrix principles and operations: (most can be illustrated in R; do it!)

1. An outer product of a vector (w/ itself) is of the form 'column vector x row vector'; in general it is a (square) matrix, but its RANK is always one (see function **`outer( )`**).

2. An inner product of, say, a MATRIX w/ itself is of the form 'row matrix x column matrix'; this product is symmetric. (The syntax for multiplication is of the form t(A) `%*%` A in R.)

3. Rank of a matrix usually refers to the number of non-zero eigenvalues (or singular values) of the matrix. (Note that in practical data analysis 'zero' is usually defined wrt some small '**`tol`**' [tolerance] threshold; *e.g.* **`tol = sqrt(.Machine$double.eps)[see ?solve`**.])

4. For a *square symmetric* matrix A, it's eigen-decomposition is written $\mathbf{Q\ D_{\lambda}^{2}\ Q'}$; if A is not symmetric, then the left and right eigenvectors are not the same, so write $\mathbf{A = U\ D_{\lambda}^{2}\ V'}$, say; if A is *rectangular, not square*, then standard notation leads to $\mathbf{A = P\ D\ Q'}$. See **`eigen`** and **`svd`** functions in R.

5. Real data matrices, and their cross products, **X'X**, or **XX'**, cannot have *negative* singular values or eigenvalues; note, however, that if the number of rows, say, *exceeds* the number of columns of **X**, then **XX'** *must have* at least one zero eigenvalue. (**`%*% is multiplication operator)`**

6. The RANK of a square (possibly symmetric) matrix cannot exceed the smallest dimension of any matrix used to compute it (in a product): Think of this w/ product like **A B C E = G**.

7. When a matrix is computed as the product of several matrices (see 5.) the rank of the product *cannot exceed the rank of the lowest rank matrix* in that product. Note that matrix multiplication is generally *not commutative*, but it is *distributive* and *associative*.

8. Matrices cannot be inverted (regular inverse) unless they are square and of *full rank.* I.e., rank must equal order of the square matrix for it to be invertible. (use **`solve( )`** for inversion)

9. Generalized inverses can be computed for (square and rectangular) matrices that are not of of full rank. There are an infinite number of *generalized inverses* for any singular matrix.

10. There is one particular generalized inverse (however) that *is unique*; it is called a Moore-Penrose inverse. It is easily computed using a singular value or eigen decomposition. For a square, symmetric matrix this inverse can be computed as $\mathbf{Q\ D_{\lambda}^{-2}\ Q'}$, with the understanding that only eigenvalues w/ magnitudes larger than '**tol**' [operational value of zero] are used in the diagonal matrix $\mathbf{D_{\lambda}^{2}}$, and only the corresponding columns of **Q** need to be used, those that correspond to the 'smallest' eigenvalues.

11. An ill-conditioned matrix is one whose columns (and rows, if square) can are 'nearly' *interdependent*, i.e., there is at least one eigenvalue that is 'near' to zero, but not quite. The term 'ill conditioned' is context dependent; that is, the term is defined in the context of the precision of the machine/algorithm used in its computation. 'Ill conditioned' in one context (like hand computation) may not be the same as 'ill-conditioned' in the context of a modern computer & refined algorithm.

12. Finally, by definition a matrix must be *complete*; *i.e.* cannot have missing values. This means that it is something of an 'oxymoron' to say that "matrix X has missing values".